ON A NONSTATIONARY FLOW PROBLEM OF A VISCOUS INCOMPRESSIBLE FLUID

(OB ODNOI NESTATSIONARNOI ZADACHE TECHENIIA VIAZKOI NESZHIMAEMOI ZHIDKOSTI)

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D.V. SHARIKADZE
(Tbilisi)

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Let us consider the flow problem of a viscous incompressible fluid between two infinite discs. The discs, separated from each other by a distance h_0, are rotating. The angular velocity of one disc is a function of time $\omega_1(t)$, and that of the other is $\omega_2(t)$. From the first disc, let there be fluid injection at a uniform time-dependent velocity $v_1(t)$, and from the second disc, at a velocity $v_2(t)$. The fluid was initially at rest. The solution of this problem in [1] was reduced to nonlinear partial differential equations which, (as was pointed out in that paper) can be solved numerically. In the present paper the solution of this problem is reduced to a system of integral equations which are solved by the method of successive approximations.

In the presence of axial symmetry and the absence of body forces, the Navier-Stokes equations in the cylindrical coordinates can be reduced to the system of nonlinear partial differential equations [1]

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial t}\right) = w\,\frac{\partial^3 w}{\partial y^3} + 4v\,\frac{\partial v}{\partial y} \tag{1}$$

$$\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial t} = w\,\frac{\partial v}{\partial y} - v\,\frac{\partial w}{\partial y}, \qquad 2u + \frac{\partial w}{\partial y} = 0 \tag{2}$$

where

$$u(y,t) = \frac{\tau_0}{r}\,V_r(r,z,\tau), \qquad v(y,t) = \frac{\tau_0}{\tau}\,V_\theta(r,z,\tau)$$
$$w(y,t) = \sqrt{\frac{\tau_0}{\nu}}\,V_z(r,z,\tau) \tag{3}$$

$$p\,\frac{\tau_0}{\rho\nu} = \frac{1}{2}\,\pi_1(t)\,\frac{r^2}{\nu\tau_0} + \pi_2(y,t), \qquad y = \frac{z}{\sqrt{\nu\tau_0}}, \qquad t = \frac{\tau}{\tau_0} \tag{4}$$

Here V_r, V_θ, V_z are radial, tangential and axial components of the velocity vector. Taking into account the boundary conditions for V_r, V_θ, V_z, determined by the nonslip condition and the presence of injection, and the initial conditions for w, v, – absence of the initial velocity – we obtain the following boundary conditions:

$$w(0,t) = w_1(t), \qquad w(h,t) = w_2(t)$$
$$\frac{\partial w(0,t)}{\partial y} = \frac{\partial w(h,t)}{\partial y} = 0, \qquad w(y,0) = 0 \qquad \left(h = \frac{h_0}{\sqrt{\nu\tau_0}}\right) \tag{5}$$

$$v(0,t) = v_1(t), \qquad v(h,t) = v_2(t), \qquad v(y,0) = 0 \tag{6}$$

We seek a solution in the form of a sum $w(y, t) = F(y, t) + \varphi(y, t)$, where $F(y, t)$ satisfies Equation (1) with zero right-hand side and the boundary conditions (5), while $\varphi(y, t)$ is a solution of Equation (2)

$$\frac{\partial^2}{\partial y^2}(\varphi_{yy} - \varphi_t) = ww_{yyy} + 4vv_y \tag{7}$$

satisfying the corresponding homogeneous limiting conditions.

Taking into account the conditions (5), the function $F(y, t)$ is sought in the form

$$F(y,t) = \psi(y,t) + \Phi(y,t) - y\Phi_y(0,t) - \Phi(0,t) \tag{8}$$

$$\psi(y,t) = \int_0^t \frac{\psi_1(\tau)}{2\sqrt{\pi}(t-\tau)^{3/2}}\, y \exp\left(-\frac{y^2}{4(t-\tau)}\right) d\tau +$$
$$+ \int_0^t \frac{\psi_2(\tau)}{\sqrt{\pi}\sqrt{t-\tau}} \exp\left(-\frac{(h-y)^2}{4(t-\tau)}\right) d\tau \tag{9}$$

We also assume that the function $\psi(y, t)$ must satisfy the conditions:

$$\psi(0,t) = w_1(t), \qquad w_y(0,t) = 0, \qquad w(y,0) = 0 \tag{10}$$

Because of these boundary conditions, the unknown functions $\psi_1(t)$ and $\psi_2(t)$ can be determined from the system of regular Volterra integral equations

$$\psi_1(t) + \int_0^t \exp\left(-\frac{y^2}{4(t-\tau)}\right) \frac{\psi_2(\tau)\,d\tau}{\sqrt{\pi}\sqrt{t-\tau}} = w_1(t)$$
$$\psi_2(t) + \int_0^t \exp\frac{-h^2}{4(t-\tau)} \frac{\psi_1(\tau)\,d\tau}{2\sqrt{\pi}(t-\tau)^{3/2}} - h^2\int_0^t \exp\frac{-h^2}{4(t-\tau)} \frac{\psi_1(\tau)\,d\tau}{4(t-\tau)^{5/2}} = 0 \tag{11}$$

The function $\Phi(y, t)$ is a solution of the heat conduction equation $\Phi_{yy} - \Phi_t = 0$, with zero initial conditions and satisfying the boundary conditions

$$\Phi(h,t)-\Phi(0,t)-h\Phi_y(0,t)=w_2(t)-\psi(h,t)=F_1(t) \tag{12}$$
$$\Phi(h,t)-\Phi(0,t)-h\Phi_y(h,t)=w_2(t)-\psi(h,t)-\psi_y(h,t)=F_2(t)$$

Let us represent the function $\Phi(y,t)$ in the form

$$\Phi(y,t)=\sqrt{\frac{1}{\pi}}\int_0^t \Phi_1(\tau)\exp\frac{-y^2}{4(t-\tau)}+\Phi_2(\tau)\exp\frac{-(y-h)^2}{4(t-\tau)}\frac{d\tau}{\sqrt{t-\tau}} \tag{13}$$

Then from the boundary conditions for $\Phi(y,t)$ we obtain for the determination of the functions $\Phi_1(t)$ and $\Phi_2(t)$ the system of regular Volterra integral equations [2]

$$\Phi_1(t)+\int_0^t[\Phi_1(\tau)K(t-\tau)+\Phi_2(\tau)L(t-\tau)]\,d\tau=F_1(t)$$
$$\Phi_2(t)+\int_0^t[\Phi_1(\tau)L(t-\tau)-\Phi_2(\tau)K(t-\tau)]\,d\tau=F_2(t) \tag{14}$$

Here

$$K(z)=\frac{1}{h}\sqrt{\frac{1}{\pi z}}\left[\exp\left(-\frac{h^2}{4z}\right)-1\right],\quad L(z)=\frac{h}{2\sqrt{\pi z^3}}\exp\left(-\frac{h^2}{4z}\right)+K(z) \tag{15}$$

For the determination of $\varphi(y,t)$ we will utilize the Green's function constructed by Dolidze [2,3,4]

$$G(y,\eta,t)=S(y,\eta,t)+g(y,\eta,t) \tag{16}$$

Here

$$S(y,\eta,t)=\frac{1}{2\sqrt{\pi t}}\int_0^y dy\int_{\alpha\eta}^{y-\eta}\exp\left(-\frac{\beta^2}{4t}\right)d\beta \tag{17}$$
$$\alpha=-1,\quad y\leqslant\mu,\qquad \alpha=1,\quad y\geqslant\mu$$

The function $g(y,\eta,t)$ is a regular solution of Equation (7) with zero right-hand side, zero initial conditions and satisfying the boundary conditions

$$g(0,\eta,t)=g_y(0,\eta,t)=0,\qquad g(h,\eta,t)=-S(h,y,t)$$
$$g_y(h,\eta,t)=-S_y(h,y,t),\qquad t>0,\quad 0<\eta<h \tag{18}$$

It can be seen, therefore, that the problem of determining g will again be reduced to the solution of a system of regular Volterra integral equations.

By the usual arguments it is easy to show the validity of the following equality [4]

$$\varphi(y,t)=\int_0^t d\tau\int_0^h (ww_{\eta\eta\eta}+4vv_\eta)\,G(y,\eta,t-\tau)\,d\eta \tag{19}$$

and, finally, we obtain for $w(y,\ t)$ the following integro-differential equation

$$w(y,t)=F(y,t)+\int_0^t d\tau\int_0^h (ww_{\eta\eta\eta}+4vv_\eta)\,G(y,\eta,t-\tau)\,d\eta \tag{20}$$

Let us determine $v(y,\ t)$. We represent the solution of the first equation in (2) as a sum of two functions

$$v(y,t)=A(y,t)+B(y,t)$$

The function $A(y,\ t)$ satisfies the first equation in (2) with zero right-hand side, and the boundary conditions (6), while the function $B(y,\ t)$ is a solution of the equation

$$\frac{\partial^2 B}{\partial y^2}-\frac{\partial B}{\partial t}=w\,\frac{\partial v}{\partial y}-v\,\frac{\partial w}{\partial y}$$

Determination of the function $v(y,\ t)$ is reduced to the following integro-differential equations [3]:

$$v(y,t)=A(y,t)+\int_0^t d\tau\int_0^h (wv_\eta-vw_\eta)\,G(y,\eta,t-\tau)\,d\eta \tag{21}$$

The function $A(y,\ t)$ is represented in the form

$$A(y,t)=\frac{1}{2\sqrt{\pi}}\int_0^t\Big[A_1(\tau)\,y\exp\Big(-\frac{y^2}{4(t-\tau)}\Big)+ A_2(\tau)(y-h)\exp\Big(-\frac{(h-y)^2}{4(t-\tau)}\Big)\Big]\frac{d\tau}{(t-\tau)^{3/2}}$$

The first two boundary conditions in (6) will be written as

$$A_1(t)-\int_0^t A_2(\tau)K(t-\tau)\,d\tau=v_1(t)\,,\qquad \int_0^t A_1(\tau)K(t-\tau)\,d\tau-A_2(t)=v_2(t) \tag{22}$$

where

$$K(z)=\frac{h}{2\sqrt{\pi z^3}}\exp\Big(-\frac{h^2}{4z}\Big)$$

Thus, the determination of $w(y,\ t)$ and $v(y,\ t)$ is reduced to a system of integro-differential equations (20) and (21). Taking (20) and (21) with a parameter δ and differentiating (20) three times and (21) once under the integral sign, which can be easily justified by (17), we obtain the following system:

$$\frac{\partial^n w}{\partial y^n} = \frac{\partial^n F}{\partial y^n} + \delta \int_0^t d\tau \int_0^h (w w_{\eta\eta\eta} + 4 v v_\eta) \frac{\partial^n G}{\partial y^n} d\eta \qquad (n = 0, 1, 3) \tag{23}$$

$$\frac{\partial^m v}{\partial y^m} = \frac{\partial^m A}{\partial y^m} + \delta \int_0^t d\tau \int_0^h (w v_\eta - v w_\eta) \frac{\partial^m G}{\partial y^m} d\eta \qquad (m = 0, 1) \tag{24}$$

The equalities (20), (23) represent a system of nonlinear integral equations for determination of the functions w, w_{yyy}, while (21), (24) are a system of nonlinear integral equations for determination of v, v_y.

We shall seek these functions in the form of series

$$\frac{\partial^n w}{\partial y^n} = \sum_{k=0}^{\infty} \delta^k \frac{\partial^n w_k}{\partial y^n}, \qquad \frac{\partial^\circ w}{\partial y^\circ} = w \qquad (n = 0, 1, 3) \tag{25}$$

$$\frac{\partial^m v}{\partial y^m} = \sum_{k=0}^{\infty} \delta^k \frac{\partial^m v_k}{\partial y^m}, \qquad \frac{\partial^\circ v}{\partial y^\circ} = v \qquad (m = 0, 1) \tag{26}$$

For determination of the terms of the series we obtain the following recurrence formulas:

$$\frac{\partial^n w_0}{\partial y^n} = \frac{\partial^n F}{\partial y^n}, \qquad \frac{\partial^m v_0}{\partial y^m} = \frac{\partial^m A}{\partial y^m} \tag{27}$$

$$\frac{\partial^n w_{k+1}}{\partial y^n} = \int_0^t d\tau \int_0^h \sum_{\alpha=0}^{k} \left(w_\alpha \frac{\partial^3 w_{k-\alpha}}{\partial \eta^3} + 4 v_\alpha \frac{\partial v_{k-\alpha}}{\partial \eta} \right) \frac{\partial^n G}{\partial y^n} d\eta \tag{28}$$

$$\frac{\partial^m v_{k+1}}{\partial y^m} = \int_0^t d\tau \int_0^h \sum_{\alpha=0}^{k} \left(w_\alpha \frac{\partial v_{k-\alpha}}{\partial \eta} - v_\alpha \frac{\partial w_{k-\alpha}}{\partial \eta} \right) \frac{\partial^m G}{\partial y^m} \partial\eta \tag{29}$$

The convergence of the series is easily proved by the method of Odqvist [5].

It is easy to show that the following inequalities take place: (M, H = const)

$$\left| \frac{\partial^n F}{\partial y^n} \right|, \qquad \left| \frac{\partial^m A}{\partial y^m} \right| < M, \qquad \int_0^t d\tau \int_0^h \left| \frac{\partial^n G}{\partial y^n} \right| d\eta \quad \int_0^t d\tau \int_0^h \left| \frac{\partial^m G}{\partial y^m} \right| d\eta < H\sqrt{t}$$

For simplicity of presentation we will show the convergence of the series (25). The proof of convergence of (26) is then easily visualized.

On the strength of (25) and (28) the majorant of the series (25) is of the form [3]

$$C = \sum_{k=0}^{\infty} \delta^k C_k \; (C_0 = M, \; C_{k+1} = 5MH\sqrt{t} \sum_{\alpha=0}^{k} C_\alpha C_{k-\alpha})$$

It is easy to verify that the equality $C = C_0 + 5MH\sqrt{(t\delta)}C^2$ is satisfied. In observing the inequality $20M^2H\delta\sqrt{t} < 1$, we have

$$C = \frac{1}{10MH\delta\sqrt{t}}\left(1 - \sqrt{1 - 20M^2H\delta\sqrt{t}}\right)$$

In this case the series (25) converge absolutely and uniformly for finite t. The absolute and uniform convergence of (26) takes place for

$$C^\circ = \frac{1}{4MH\delta\sqrt{t}}\left(1 - \sqrt{1 - 8M^2H\delta\sqrt{t}}\right)$$

and since the derived solutions are obtained from (20) and (21) for $\delta = 1$, then the series (25) and (26) will yield a solution if $20M^2H\sqrt{t} < 1$

In order to find the pressure it is necessary to find the functions $\pi_1(t)$ and $\pi_2(y, t)$ which are determined after solution of the problem (1) to (6) from the equations

$$\pi_1(t) = \frac{\partial^2 u}{\partial y^2} + v^2 - u^2 - w\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t}, \quad \frac{\partial \pi_2(t)}{\partial y} = \frac{\partial^2 w}{\partial y^2} - w\frac{\partial w}{\partial y} - \frac{\partial w}{\partial t}$$

These equations are also valid for the discs with a finite radius if the radius R is large compared to the distance R_0 between the discs. For discs with a finite radius $R \gg R_0$, one can find the retarding torques [1]

$$M_0(t) = \frac{\pi\rho R^4}{2}\sqrt{\frac{\nu}{\tau_0^3}}\frac{\partial v(0, t)}{\partial y}, \quad M_1(t) = \frac{\pi\rho R^4}{2}\sqrt{\frac{\nu}{\tau_0^3}}\frac{\partial v(h, t)}{\partial y}$$

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